Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics
Higher level
Paper 1

Wednesday 11 November 2015 (morning)

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all questions in the boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics HL and further mathematics HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[120 marks]**.

13 pages

8815 – 7201

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

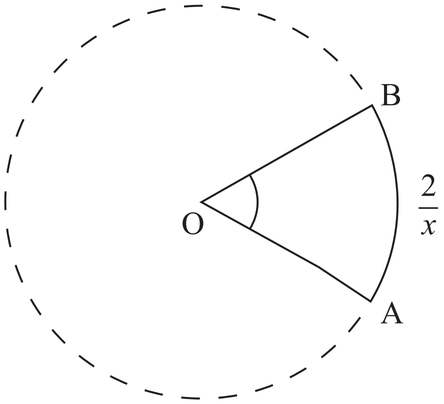
Section A

Answer **all** questions in the boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 4]

The following diagram shows a sector of a circle where $\mathrm{A\hat{O}B} = x$ radians and the length of the arc $\mathrm{AB} = \frac{2}{x}\,\mathrm{cm}$.

Given that the area of the sector is $16\,\mathrm{cm}^2$, find the length of the arc AB.



2. [Maximum mark: 4]

Using integration by parts find $\int x \sin x \, dx$.

3. [Maximum mark: 6]

(a) Write down and simplify the expansion of $(2+x)^4$ in ascending powers of x. [3]

(b) Hence find the exact value of $(2.1)^4$. [3]

4. [Maximum mark: 6]

Consider the curve $y = \frac{1}{1-x}$, $x \in \mathbb{R}$, $x \neq 1$.

(a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$. [2]

(b) Determine the equation of the normal to the curve at the point $x = 3$ in the form $ax + by + c = 0$ where a, b, $c \in \mathbb{Z}$. [4]

5. [Maximum mark: 4]

Use the substitution $u = \ln x$ to find the value of $\int_{e}^{e^2} \frac{dx}{x \ln x}$.

6. [Maximum mark: 7]

A box contains four red balls and two white balls. Darren and Marty play a game by each taking it in turn to take a ball from the box, without replacement. The first player to take a white ball is the winner.

(a) Darren plays first, find the probability that he wins. [4]

The game is now changed so that the ball chosen is replaced after each turn.
Darren still plays first.

(b) Show that the probability of Darren winning has not changed. [3]

7. [Maximum mark: 8]

A curve is defined by $xy = y^2 + 4$.

(a) Show that there is no point where the tangent to the curve is horizontal. [4]

(b) Find the coordinates of the points where the tangent to the curve is vertical. [4]

8. [Maximum mark: 8]

(a) Show that $\sin\left(\theta + \frac{\pi}{2}\right) = \cos\theta$. [1]

(b) Consider $f(x) = \sin(ax)$ where a is a constant. Prove by mathematical induction that $f^{(n)}(x) = a^n \sin\left(ax + \frac{n\pi}{2}\right)$ where $n \in \mathbb{Z}^+$ and $f^{(n)}(x)$ represents the n^{th} derivative of $f(x)$. [7]

9. [Maximum mark: 7]

Solve the equation $\sin 2x - \cos 2x = 1 + \sin x - \cos x$ for $x \in [-\pi, \pi]$.

10. [Maximum mark: 6]

A given polynomial function is defined as $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$. The roots of the polynomial equation $f(x) = 0$ are consecutive terms of a geometric sequence with a common ratio of $\frac{1}{2}$ and first term 2.

Given that $a_{n-1} = -63$ and $a_n = 16$ find

(a) the degree of the polynomial; [4]

(b) the value of a_0. [2]

Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 17]

(a) Solve the equation $z^3 = 8\mathrm{i}$, $z \in \mathbb{C}$ giving your answers in the form $z = r(\cos\theta + \mathrm{i}\sin\theta)$ **and** in the form $z = a + b\mathrm{i}$ where $a, b \in \mathbb{R}$. [6]

(b) Consider the complex numbers $z_1 = 1 + \mathrm{i}$ and $z_2 = 2\left(\cos\left(\frac{\pi}{6}\right) + \mathrm{i}\sin\left(\frac{\pi}{6}\right)\right)$.

(i) Write z_1 in the form $r(\cos\theta + \mathrm{i}\sin\theta)$.

(ii) Calculate $z_1 z_2$ and write in the form $z = a + b\mathrm{i}$ where $a, b \in \mathbb{R}$.

(iii) Hence find the value of $\tan\frac{5\pi}{12}$ in the form $c + d\sqrt{3}$, where $c, d \in \mathbb{Z}$.

(iv) Find the smallest value $p > 0$ such that $(z_2)^p$ is a positive real number. [11]

12. [Maximum mark: 20]

Consider the function defined by $f(x) = x\sqrt{1-x^2}$ on the domain $-1 \le x \le 1$.

(a) Show that f is an odd function. [2]

(b) Find $f'(x)$. [3]

(c) Hence find the x-coordinates of any local maximum or minimum points. [3]

(d) Find the range of f. [3]

(e) Sketch the graph of $y = f(x)$ indicating clearly the coordinates of the x-intercepts and any local maximum or minimum points. [3]

(f) Find the area of the region enclosed by the graph of $y = f(x)$ and the x-axis for $x \ge 0$. [4]

(g) Show that $\int_{-1}^{1}\left|x\sqrt{1-x^2}\right|\mathrm{d}x > \left|\int_{-1}^{1} x\sqrt{1-x^2}\,\mathrm{d}x\right|$. [2]

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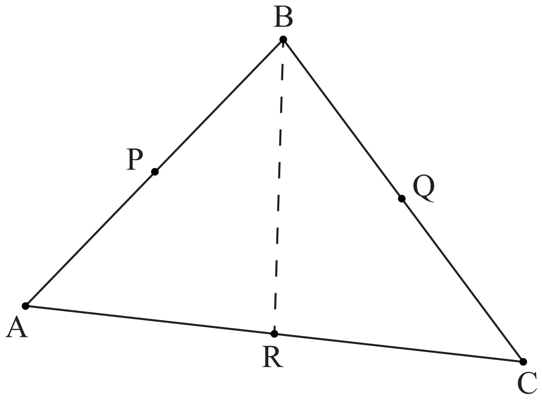
13. [Maximum mark: 23]



Consider the triangle ABC. The points P, Q and R are the midpoints of the line segments [AB], [BC] and [AC] respectively.

Let $\overrightarrow{OA} = \boldsymbol{a}$, $\overrightarrow{OB} = \boldsymbol{b}$ and $\overrightarrow{OC} = \boldsymbol{c}$.

(a) Find $\overrightarrow{BR}$ in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$. [2]

(b) (i) Find a vector equation of the line that passes through B and R in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ and a parameter λ.

(ii) Find a vector equation of the line that passes through A and Q in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ and a parameter μ.

(iii) Hence show that $\overrightarrow{OG} = \frac{1}{3}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})$ given that G is the point where [BR] and [AQ] intersect. [9]

(c) Show that the line segment [CP] also includes the point G. [3]

The coordinates of the points A, B and C are $(1, 3, 1)$, $(3, 7, -5)$ and $(2, 2, 1)$ respectively.
A point X is such that [GX] is perpendicular to the plane ABC.

(d) Given that the tetrahedron ABCX has volume 12 units^3, find possible coordinates of X. [9]

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